Solving quadratic equations by factorisation

A LEVEL LINKS
Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose products is ac.
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

Examples

Example 1 Solve $5x^2 = 15x$

$5x^2 = 15x$	**1**	Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by x as this would lose the solution $x = 0$.
$5x^2 - 15x = 0$		
$5x(x - 3) = 0$	**2**	Factorise the quadratic equation. $5x$ is a common factor.
So $5x = 0$ or $(x - 3) = 0$	**3**	When two values multiply to make zero, at least one of the values must be zero.
Therefore $x = 0$ or $x = 3$	**4**	Solve these two equations.

Example 2 Solve $x^2 + 7x + 12 = 0$

$x^2 + 7x + 12 = 0$	**1**	Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3)
$b = 7$, $ac = 12$		
$x^2 + 4x + 3x + 12 = 0$	**2**	Rewrite the b term ($7x$) using these two factors.
$x(x + 4) + 3(x + 4) = 0$	**3**	Factorise the first two terms and the last two terms.
$(x + 4)(x + 3) = 0$	**4**	$(x + 4)$ is a factor of both terms.
So $(x + 4) = 0$ or $(x + 3) = 0$	**5**	When two values multiply to make zero, at least one of the values must be zero.
Therefore $x = -4$ or $x = -3$	**6**	Solve these two equations.

Example 3   Solve $9x^2 - 16 = 0$

$9x^2 - 16 = 0$ $(3x + 4)(3x - 4) = 0$ So $(3x + 4) = 0$ or $(3x - 4) = 0$ $x = -\dfrac{4}{3}$ or $x = \dfrac{4}{3}$	**1** Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$. **2** When two values multiply to make zero, at least one of the values must be zero. **3** Solve these two equations.

Example 4   Solve $2x^2 - 5x - 12 = 0$

$b = -5$, $ac = -24$ So $2x^2 - 8x + 3x - 12 = 0$ $2x(x - 4) + 3(x - 4) = 0$ $(x - 4)(2x + 3) = 0$ So $(x - 4) = 0$ or $(2x + 3) = 0$ $x = 4$ or $x = -\dfrac{3}{2}$	**1** Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. $(-8$ and $3)$ **2** Rewrite the b term $(-5x)$ using these two factors. **3** Factorise the first two terms and the last two terms. **4** $(x - 4)$ is a factor of both terms. **5** When two values multiply to make zero, at least one of the values must be zero. **6** Solve these two equations.

Practice

1   Solve

    a $6x^2 + 4x = 0$ **b** $28x^2 - 21x = 0$

    c $x^2 + 7x + 10 = 0$ **d** $x^2 - 5x + 6 = 0$

    e $x^2 - 3x - 4 = 0$ **f** $x^2 + 3x - 10 = 0$

    g $x^2 - 10x + 24 = 0$ **h** $x^2 - 36 = 0$

    i $x^2 + 3x - 28 = 0$ **j** $x^2 - 6x + 9 = 0$

    k $2x^2 - 7x - 4 = 0$ **l** $3x^2 - 13x - 10 = 0$

2   Solve

    a $x^2 - 3x = 10$ **b** $x^2 - 3 = 2x$

    c $x^2 + 5x = 24$ **d** $x^2 - 42 = x$

    e $x(x + 2) = 2x + 25$ **f** $x^2 - 30 = 3x - 2$

    g $x(3x + 1) = x^2 + 15$ **h** $3x(x - 1) = 2(x + 1)$

> **Hint**
>
> Get all terms onto one side of the equation.

Solving quadratic equations by completing the square

Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

Examples

Example 5 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$x^2 + 6x + 4 = 0$	**1** Write $x^2 + bx + c = 0$ in the form
$(x + 3)^2 - 9 + 4 = 0$	$\left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2 + c = 0$
$(x + 3)^2 - 5 = 0$	**2** Simplify.
$(x + 3)^2 = 5$	**3** Rearrange the equation to work out x. First, add 5 to both sides.
$x + 3 = \pm\sqrt{5}$	**4** Square root both sides. Remember that the square root of a value gives two answers.
$x = \pm\sqrt{5} - 3$	**5** Subtract 3 from both sides to solve the equation.
So $x = -\sqrt{5} - 3$ or $x = \sqrt{5} - 3$	**6** Write down both solutions.

Example 6 Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

$2x^2 - 7x + 4 = 0$	**1** Before completing the square write $ax^2 + bx + c$ in the form
$2\left(x^2 - \dfrac{7}{2}x\right) + 4 = 0$	$a\left(x^2 + \dfrac{b}{a}x\right) + c$
$2\left[\left(x - \dfrac{7}{4}\right)^2 - \left(\dfrac{7}{4}\right)^2\right] + 4 = 0$	**2** Now complete the square by writing $x^2 - \dfrac{7}{2}x$ in the form $\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{b}{2a}\right)^2$
$2\left(x - \dfrac{7}{4}\right)^2 - \dfrac{49}{8} + 4 = 0$	**3** Expand the square brackets.
$2\left(x - \dfrac{7}{4}\right)^2 - \dfrac{17}{8} = 0$	**4** Simplify.
	(continued on next page)

$2\left(x-\dfrac{7}{4}\right)^2=\dfrac{17}{8}$	**5** Rearrange the equation to work out x. First, add $\dfrac{17}{8}$ to both sides.
$\left(x-\dfrac{7}{4}\right)^2=\dfrac{17}{16}$	**6** Divide both sides by 2.
$x-\dfrac{7}{4}=\pm\dfrac{\sqrt{17}}{4}$	**7** Square root both sides. Remember that the square root of a value gives two answers.
$x=\pm\dfrac{\sqrt{17}}{4}+\dfrac{7}{4}$	**8** Add $\dfrac{7}{4}$ to both sides.
So $x=\dfrac{7}{4}-\dfrac{\sqrt{17}}{4}$ or $x=\dfrac{7}{4}+\dfrac{\sqrt{17}}{4}$	**9** Write down both the solutions.

Practice

3 Solve by completing the square.

    a $x^2-4x-3=0$ **b** $x^2-10x+4=0$

    c $x^2+8x-5=0$ **d** $x^2-2x-6=0$

    e $2x^2+8x-5=0$ **f** $5x^2+3x-4=0$

4 Solve by completing the square.

    a $(x-4)(x+2)=5$

    b $2x^2+6x-7=0$

    c $x^2-5x+3=0$

> **Hint**
>
> Get all terms onto one side of the equation.

Solving quadratic equations by using the formula

Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for a, b and c.

Examples

Example 7   Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$a = 1$, $b = 6$, $c = 4$ $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$	**1** Identify a, b and c and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$$	**2** Substitute $a = 1$, $b = 6$, $c = 4$ into the formula.
$$x = \frac{-6 \pm \sqrt{20}}{2}$$	**3** Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2.
$$x = \frac{-6 \pm 2\sqrt{5}}{2}$$	**4** Simplify $\sqrt{20}$. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
$x = -3 \pm \sqrt{5}$	**5** Simplify by dividing numerator and denominator by 2.
So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$	**6** Write down both the solutions.

Example 8 Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

$a = 3, b = -7, c = -2$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**1** Identify a, b and c, making sure you get the signs right and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$	**2** Substitute $a = 3$, $b = -7$, $c = -2$ into the formula.
$x = \dfrac{7 \pm \sqrt{73}}{6}$ So $x = \dfrac{7 - \sqrt{73}}{6}$ or $x = \dfrac{7 + \sqrt{73}}{6}$	**3** Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2. **4** Write down both the solutions.

Practice

5 Solve, giving your solutions in surd form.
   a $3x^2 + 6x + 2 = 0$ **b** $2x^2 - 4x - 7 = 0$

6 Solve the equation $x^2 - 7x + 2 = 0$

   Give your solutions in the form $\dfrac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

7 Solve $10x^2 + 3x + 3 = 5$
   Give your solution in surd form.

> **Hint**
>
> Get all terms onto one side of the equation.

Extend

8 Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.
   a $4x(x - 1) = 3x - 2$
   b $10 = (x + 1)^2$
   c $x(3x - 1) = 10$

Answers

1 **a** $x = 0$ or $x = -\dfrac{2}{3}$ **b** $x = 0$ or $x = \dfrac{3}{4}$

     c $x = -5$ or $x = -2$ **d** $x = 2$ or $x = 3$

     e $x = -1$ or $x = 4$ **f** $x = -5$ or $x = 2$

     g $x = 4$ or $x = 6$ **h** $x = -6$ or $x = 6$

     i $x = -7$ or $x = 4$ **j** $x = 3$

     k $x = -\dfrac{1}{2}$ or $x = 4$ **l** $x = -\dfrac{2}{3}$ or $x = 5$

2 **a** $x = -2$ or $x = 5$ **b** $x = -1$ or $x = 3$

     c $x = -8$ or $x = 3$ **d** $x = -6$ or $x = 7$

     e $x = -5$ or $x = 5$ **f** $x = -4$ or $x = 7$

     g $x = -3$ or $x = 2\dfrac{1}{2}$ **h** $x = -\dfrac{1}{3}$ or $x = 2$

3 **a** $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$ **b** $x = 5 + \sqrt{21}$ or $x = 5 - \sqrt{21}$

     c $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$ **d** $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$

     e $x = -2 + \sqrt{6.5}$ or $x = -2 - \sqrt{6.5}$ **f** $x = \dfrac{-3 + \sqrt{89}}{10}$ or $x = \dfrac{-3 - \sqrt{89}}{10}$

4 **a** $x = 1 + \sqrt{14}$ or $x = 1 - \sqrt{14}$ **b** $x = \dfrac{-3 + \sqrt{23}}{2}$ or $x = \dfrac{-3 - \sqrt{23}}{2}$

     c $x = \dfrac{5 + \sqrt{13}}{2}$ or $x = \dfrac{5 - \sqrt{13}}{2}$

5 **a** $x = -1 + \dfrac{\sqrt{3}}{3}$ or $x = -1 - \dfrac{\sqrt{3}}{3}$ **b** $x = 1 + \dfrac{3\sqrt{2}}{2}$ or $x = 1 - \dfrac{3\sqrt{2}}{2}$

6 $x = \dfrac{7 + \sqrt{41}}{2}$ or $x = \dfrac{7 - \sqrt{41}}{2}$

7 $x = \dfrac{-3 + \sqrt{89}}{20}$ or $x = \dfrac{-3 - \sqrt{89}}{20}$

8 **a** $x = \dfrac{7 + \sqrt{17}}{8}$ or $x = \dfrac{7 - \sqrt{17}}{8}$

     b $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$

     c $x = -1\dfrac{2}{3}$ or $x = 2$